Comencem

- **Escriu l'expressió algèbrica de cinc funcions que tinguin per derivada la funció $f(x) = 2x + 3$.**

  Resposta oberta. Per exemple:

  $F_1(x) = x^2 + 3x$; $F_2(x) = x^2 + 3x + 1$;

  $F_3(x) = x^2 + 3x + 10$; $F_4(x) = x^2 + 3x - \sqrt{-2}$;
  $F_5(x) = x^2 + 3x - \pi$

- **Se sap que la derivada d'una funció $G(x)$ és $g(x) = e^x$. Si la gràfica de la funció $G(x)$ passa pel punt (0,3), quina de les funcions següents és $G(x)$?**

  ***a*) $G(x) = e^x + 3$ *b*) $G(x) = e^x + 2$ *c*) $G(x) = e^x - 3$**

  $G(x) = e^x + 3$, ja que $G(0) = 3$.

- **Escriu l'equació de tres funcions que tinguin per derivada la funció $f(x) = 2$. Representa-les gràficament i comprova que pots obtenir la gràfica de cadascuna d'aquestes funcions per translació d'una qualsevol de les altres dues.**

  Resposta oberta. Per exemple:

  $F_1(x) = 2x$; $F_2(x) = 2x + 3$; $F_3(x) = 2x - 2$;

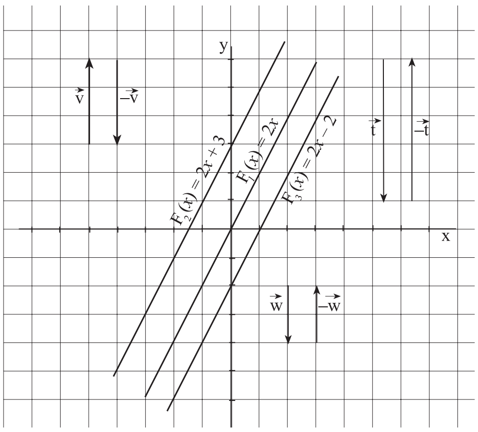


Figura 5.1

El vector $\vec{v} = (0, 3)$ permet passar de la gràfica de $F_1(x)$ a la de $F_2(x)$, i el vector $-\vec{v}$, de la gràfica $F_2(x)$ a la de $F_1(x)$. El vector $\vec{w} = (0, -2)$ trasllada la gràfica de $F_1(x)$ a la de $F_3(x)$, i el vector $-\vec{w}$, la gràfica de $F_3(x)$ a $F_1(x)$. Finalment, el vector $\vec{t} = (0, -5)$, permet passar de la gràfica de $F_2(x)$ a la de $F_3(x)$, e el vector $-\vec{t}$, de la gràfica de $F_3(x)$ a la de $F_2(x)$.

Exercicis

1. Escriu l'expressió general de les primitives de cadascuna de les funcions següents:

***a*) $f(x) = 3x^2$**

$F(x) = x^3 + C$

***b*) $g(x) = \sin x$**

$G(x) = -\cos x + C$

***c*) $h(x) = -5$**

$H(x) = -5x + C$

***d*) $i(x) = \dfrac{1}{x}$**

$I(x) = \ln x + C$

2. Determina la funció primitiva de la funció:

$f(x) = \cos x$

la gràfica de la qual passi pel punt de coordenades .

$F(x) = \sin x + C$

$$F\left(\frac{\pi}{2}\right) = 4 \rightarrow 4 = \sin\frac{\pi}{2} + C \rightarrow 4 = 1 + C \rightarrow C = 3$$

$F(x) = \sin x + 3$

3. Se sap que la funció:

$$F(x) = \frac{x^2 + 1}{x^2 - 1}$$

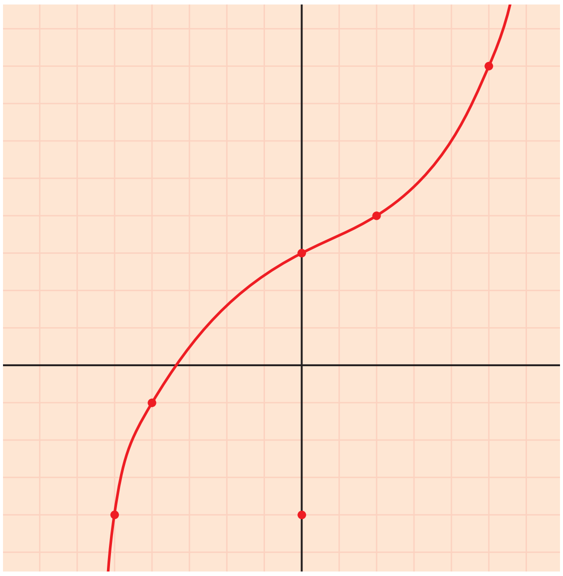
és una primitiva de la funció f(x). Quina és la funció f(x)?

$$f(x) = F'(x) = \frac{2x(x^2 - 1) - (x^2 + 1)\cdot 2x}{(x^2 - 1)^2} = \frac{-4x}{(x^2 - 1)^2}$$

4. Comprova que totes les primitives de la funció $f(x) = \ln x$ són del tipus $F(x) = x(\ln x - 1) + C$.

$F'(x) = \ln x - 1 + x\,\dfrac{1}{x} = \ln x + 1 - 1 = \ln x = f(x)$

5. Si G_1 i G_2 són dues primitives d'una mateixa funció g, es poden tallar els seus gràfics? Dibuixa la gràfica de la funció G_1 sabent que passa pel punt (0, –4) si la gràfica de la funció G_2 és el de la figura 5.4.



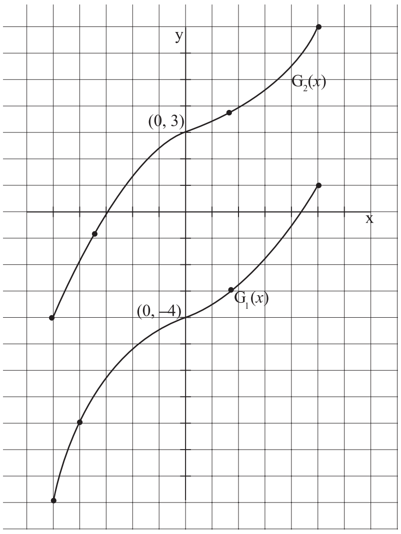


Fig. 5.2

No es poden tallar, ja que les expressions algebraiques de les funcions $G_1(x)$ i $G_2(x)$ només es diferencien en una constant.

6. Calcula la derivada de les funcions següents i escriu-ne després les corresponents integrals indefinides:

a) ***f*(*x*) = tg *x***

$$f'(x) = \frac{1}{\cos^2 x} = 1 + tg^2 x \rightarrow \int \frac{1}{\cos^2 x} dx =$$

$$\int (1 + tg^2 x) dx = tgx + C$$

b) ***g*(*x*) = 2^{3x+5}**

$$g'(x) = 2^{3x+5} 3 \ln 2 \rightarrow \int 2^{3x+5} 3 \ln 2 dx = 2^{3x+5} + C$$

c) $\boldsymbol{h(x) = \dfrac{x^2}{x^2 - 4}}$

$$h'(x) = \frac{2x(x^2-4) - x^2 2x}{(x^2-4)^2} = \frac{-8x}{(x^2-4)^2} \rightarrow$$

$$\rightarrow \int \frac{-8x}{(x^2-4)^2} dx = \frac{x^2}{x^2-4} + C$$

d) ***i*(*x*) = ln² *x***

$$i'(x) = 2 \ln x \frac{1}{x} = \frac{2 \ln x}{x} \rightarrow \int \frac{2 \ln x}{x} dx = \ln^2 x + C$$

7. Troba la derivada de les funcions següents:

a) ***f*(*x*) = ∫*x* 3*ˣ* dx**

$f'(x) = x3^x$

b) ***g*(*x*) = ∫cos² *x* dx**

$g'(x) = \cos^2 x$

c) ***h*(*x*) = ∫(tg *x* – ln *x*) dx**

$h'(x) = \text{tg } x - \ln x$

d) ***i*(*x*) = ∫*x*² eˣ dx**

$i'(x) = x^2 e^x$

8. Un mòbil recorre una trajectòria rectilínia amb una acceleració constant de 2 m/s². Se sap que en el moment de començar a comptar el temps, *v*(0) = 3 m/s i *s*(0) = –5 m.

Troba les expressions de les funcions *v* = *v*(*t*) i *s* = *s*(*t*) corresponents al seu moviment.

Cal que recordis:

$$s = s(t) \xrightarrow{\text{derivant}} v = v(t) \xrightarrow{\text{derivant}} a = a(t)$$

$$v(t) = \int 2dt = 2t + C$$

$v(0) = 3$ m/s $\rightarrow 3 = 2 \cdot 0 + C \rightarrow C = 3$ m/s

$v(t) = 2t + 3$ m/s

$$s(t) = \int (2t + 3)dt = t^2 + 3t + C'$$

$s(0) = -5$ m $\rightarrow -5 = 0^2 + 3 \cdot 0 + C \rightarrow C' = -5$ m

$s(t) = t^2 + 3t - 5$ m

9. Comprova que les derivades de les funcions següents:

$$F(x) = \frac{x^{n+1}}{n+1},\ n \in \mathbb{R},\ n \neq -1 \text{ i } G(x) = \frac{a^x}{\ln a}$$

són, respectivament, *f*(*x*) = *xⁿ* i *g*(*x*) = *aˣ*.

$$F'(x) = \frac{1}{n+1} \cdot (n+1)x^n = x^n = f(x)$$

$$G'(x) = \frac{1}{\ln a} \cdot a^x \cdot \ln a = a^x = g(x)$$

10. **Troba $\int x^{-1}\,dx$.**

$$\int x^{-1}dx = \int \frac{1}{x}dx = \ln|x| + C$$

11. **Calcula les primitives següents:**

a) $\int \frac{1}{x^4}dx$

$$\int \frac{1}{x^4}dx = \int x^{-4}dx = \frac{x^{-3}}{-3} + C = \frac{-1}{3x^3} + C$$

b) $\int \sqrt[4]{x^3}\,dx$

$$\int \sqrt[4]{x^3}\,dx = \int x^{3/4}dx = \frac{x^{7/4}}{7/4} + C = \frac{4}{7}\sqrt[4]{x^7} + C$$

c) $\int \frac{\sqrt[3]{x}}{x^2}dx$

$$\int \frac{\sqrt[3]{x}}{x^2}dx = \int \frac{x^{1/2}}{x^2}dx = \int x^{-5/3}dx = \frac{x^{-2/3}}{-2/3}dx = \frac{-3}{2\sqrt[3]{x^2}} + C$$

d) $\int \frac{3^x}{4^x}dx$

$$\int \frac{3^4}{4^x}dx = \int \left(\frac{3}{4}\right)^x dx = \frac{(\frac{3}{4})^x}{\ln\frac{3}{4}} + C$$

12. **Determina la primitiva de la funció $f(x) = 1 + \mathrm{tg}^2\, x$ la gràfica de la qual conté el punt $\left(\frac{\pi}{4}, 3\right)$.**

$$F(x) = \int (1 + \mathrm{tg}^2 x)dx = \mathrm{tg}x + C$$

$$F\left(\frac{\pi}{4}\right) = 3 \rightarrow 3 = \mathrm{tg}\frac{\pi}{4} + C \rightarrow 3 = 1 + C \rightarrow C = 2$$

$$F(x) = \mathrm{tg}x + 2$$

13. **Calcula:**

a) $\int 2x\sqrt{1 + x^2}\,dx$

$$\int 2x\sqrt{1 + x^2}\,dx = \int 2x(1 + x^2)^{1/2}dx =$$

$$= \frac{(1 + x^2)^{3/2}}{3/2} + C = \frac{2}{3}\sqrt{(1 + x^2)^3 + C}$$

b) $\int -\sin x \cos^2 x\, dx$

$$\int -\sin x \cos^2 x dx = \frac{\cos^3 x}{3} + C$$

c) $\int \frac{\text{arc tgx}}{1 + x^2}dx$

$$\int \frac{\mathrm{arctg}x}{1 + x^2}dx = \int \mathrm{arctg}x \cdot \frac{1}{1 + x^2} = \frac{(\mathrm{arctg}x^2)}{2} + C$$

d) $\int \frac{1}{(x - 3)^2}dx$

$$\int \frac{1}{(x - 3)^2}dx = \int (x - 3)^{-2}dx = \frac{(x - 3)^{-1}}{-1} + C =$$

$$= \frac{-1}{x - 3} + C$$

e) $\int \frac{2x + 1}{x^2 + x - 10}dx$

$$\int \frac{2x + 1}{x^2 + x - 10}dx = \ln|x^2 + x - 10| + C$$

f) $\int \frac{2\sin x \cos x}{1 + \sin^2 x}dx$

$$\int \frac{2\sin x \cos x}{1 + \sin^2 x}dx = \ln(1 + \sin^2 x) + C$$

14. **Troba la primitiva de la funció $f(x) = \sin x \cos x$ la gràfica de la qual passa pel punt $\left(\frac{\pi}{2}, \frac{15}{2}\right)$.**

$$F(x) = \int \sin x \cos x dx = \frac{\sin^2 x}{2} + C$$

$$F\left(\frac{\pi}{2}\right) = \frac{15}{2} \rightarrow \frac{15}{2} = \frac{\sin^2\left(\frac{\pi}{2}\right)}{2} + C \rightarrow$$

$$\rightarrow \frac{15}{2} = \frac{1}{2} + C \rightarrow C = 7$$

$$F(x) = \frac{\sin^2 x}{2} + 7$$

15. **Justifica el motiu pel qual podem afirmar que no hi ha cap primitiva de la funció $f(x) = \frac{1}{(x - 2)^2}$ que presenti màxims ni mínims relatius en el seu domini.**

Sigui $F(x)$ una primitiva de $f(x)$.

$$F'(x) = f(x) = \frac{1}{(x - 2)^2}$$

Per trobar els màxims i mínims relatius de $F(x)$ cal resoldre l'equació $F'(x) = 0$. És senzill observar que aquesta equació no té solució.

16. Troba la primitiva de la funció $f(x) = -\sin x\, e^{\cos x}$ la gràfica de la qual talla l'eix d'abscisses en $x = \frac{\pi}{2}$.

$$F(x) = \int -\sin x e^{\cos x} dx = e^{\cos x} + C$$

$$F\left(\frac{\pi}{2}\right) = 0 \rightarrow 0 = e^{\cos\frac{\pi}{2}} + C \rightarrow 0 = 1 + C \rightarrow C = -1$$

$$F(x) = e^{\cos x} - 1$$

17. Calcula:

a) $\int 4x^3 \sin(x^4 - 3)\, dx$

$$\int 4x^3 \sin(x^4 - 3) dx = -\cos(x^4 - 3) + C$$

b) $\int \frac{e^{tgx}}{\cos^2 x}\, dx$

$$\int \frac{e^{tgx}}{\cos^2 x} dx = \int e^{tgx} \cdot \frac{1}{\cos^2 x} dx = e^{tgx} + C$$

c) $\int \frac{4^{\ln x}}{x}\, dx$

$$\int \frac{4^{\ln x}}{x} dx = \int 4^{\ln x} \cdot \frac{1}{x} dx = \frac{4^{\ln x}}{\ln 4} + C$$

d) $\int \frac{2x}{1 + x^4}\, dx$

$$\int \frac{2x}{1+x^4} dx = \int \frac{2x}{1+(x^2)^2} dx = \text{arctg} x^2 + C$$

e) $\int \frac{1 + tg^2 x}{tgx}\, dx$

$$\int \frac{1 + \text{tg}^2 x}{\text{tg} x} dx = \ln|\text{tg} x| + C$$

f) $\int (\text{tg}^2 x + \text{tg}^4 x)\, dx$

$$\int (\text{tg}^2 x + \text{tg}^4 x) dx = \int \text{tg}^2 x(1 + \text{tg}^2 x) dx = \frac{\text{tg}^3 x}{3} + C$$

$$= \frac{\text{tg}^3 x}{3} + C$$

18. Calcula:

a) $\int (3x^2 - 1) \cos(x^3 - x)\, dx$

$$\int (3x^2 - 1)\cos(x^3 - x) dx = \sin(x^3 - x) + C$$

b) $\int \frac{2x}{\sqrt{1 + x^2}}\, dx$

$$\int \frac{2x}{\sqrt{1+x^2}} dx = \int 2x \cdot (1 + x^2)^{-1/2} dx =$$

$$= \frac{(1+x^2)^{1/2}}{1/2} + C = 2\sqrt{1+x^2} + C$$

c) $\int 3x^2 \sin x^3\, dx$

$$\int 3x^2 \sin x^3 dx = -\cos x^3 + C$$

d) $\int \frac{e^x}{e^x + 9}\, dx$

$$\int \frac{e^x}{e^x + 9} dx = \ln(e^x + 9) + C$$

e) $\int \frac{1}{\sqrt{1 - x^2} \arcsin x}\, dx$

$$\int \frac{1}{\sqrt{1-x^2} \arcsin x} dx = \int \frac{\frac{1}{\sqrt{1-x^2}}}{\arcsin x} dx =$$

$$= \ln|\arcsin x| + C$$

f) $\int \frac{\cos\sqrt{x}}{2\sqrt{x}}\, dx$

$$\int \frac{\cos\sqrt{x}}{2\sqrt{x}} dx = \int \cos\sqrt{x} \cdot \frac{1}{2\sqrt{x}} dx = \sin\sqrt{x} + C$$

19. Determina les asímptotes de la funció:

$F(x) = \int \frac{1}{(x+3)^2}\, dx$ **sabent que $F(-2) = 2$**

$$F(x) = \int \frac{1}{(x+3)^2} dx = \int (x+3)^{-2} dx =$$

$$= \frac{(x+3)^{-1}}{-1} + C = \frac{-1}{(x+3)} + C$$

$$F(-2) = 2 \rightarrow \frac{-1}{-2+3} + C = 2 \rightarrow C = 3$$

$$F(x) = \frac{-1}{x+3} + 3 = \frac{-1 + 3x + 9}{x+3} = \frac{3x+8}{x+3}$$

$$x + 3 = 0 \rightarrow x = -3$$

Asímptota vertical: la recta $x = -3$

$$\lim_{x\to\infty} F(x) = \lim_{x\to\infty} \frac{3x+8}{x+3} = 3$$

Asímptota horitzontal: la recta $y = 3$

20. Calcula:

a) **∫(2x^3 – 3x^2 + 5x – 1) dx**

$$\int(2x^3-3x^2+5x-1)dx=$$
$$=\frac{1}{2}x^4-x^3+\frac{5}{2}x^2-x+C$$

b) $\int\frac{2x+5}{7x}\,dx$

$$\int\frac{2x+5}{7x}dx=\int\left(\frac{2}{7}+\frac{5}{7}\cdot\frac{1}{x}\right)dx=\frac{2}{7}x+$$
$$+\frac{5}{7}\ln|x|+C$$

c) **∫(3^{2x} – e^{4x} + 1) dx**

$$\int(3^{2x}-e^{4x}+1)dx=$$
$$\frac{1}{2\ln 3}\int 2\ln 3\cdot 3^{2x}dx-\frac{1}{4}\int 4\cdot e^{4x}dx+\int dx=$$
$$=\frac{3^{2x}}{2\ln 3}-\frac{1}{4}e^{4x}+x+C$$

d) $\int\frac{5}{(2x-1)^2}\,dx$

$$\int\frac{5}{(2x-1)^2}dx=\frac{5}{2}\int 2\cdot(2x-1)^{-2}dx=$$
$$=\frac{5}{2}\cdot\frac{(2x-1)^{-1}}{-1}+C=\frac{-5}{2(2x-1)}+C$$

e) **∫(2x – 3)(2x + 3) dx**

$$\int(2x-3)(2x+3)dx=\int(4x^2-9)dx=$$
$$=\frac{4}{3}x^3-9x+C$$

f) $\int\frac{9}{7x+3}\,dx$

$$\int\frac{9}{(7x+3)}dx=\frac{9}{7}\int\frac{7}{7x+3}dx=$$
$$=\frac{9}{7}\ln|7x+3|+C$$

g) $\int\frac{2x^3-x^2}{3x^2}\,dx$

$$\int\frac{2x^3-x^2}{3x^2}dx=\int\left(\frac{2}{3}x-\frac{1}{3}\right)dx=$$
$$=\frac{1}{3}(x^2-x)+C$$

h) $\int\frac{7x}{5x^2-3}\,dx$

$$\int\frac{7x}{5x^2-3}dx=\frac{7}{10}\int\frac{10x}{5x^2-3}dx=$$
$$\frac{7}{10}\ln\left|5x^2-3\right|+C$$

21. Se sap que la gràfica d'una funció passa pel punt *P*(1, 4) i que el pendent de la recta tangent en qualsevol punt d'aquesta gràfica s'expressa mitjançant $m(x)$ = 2x^2 – 3x + 5. Determina l'expressió algèbrica d'aquesta funció.

$$F(x)=\int(2x^2-3x+5)dx=\frac{2}{3}x^3-\frac{3}{2}x^2+5x+C$$
$$F(1)=4\rightarrow\frac{2}{3}-\frac{3}{2}+5+C=4\rightarrow C=-\frac{1}{6}$$
$$F(x)=\frac{2}{3}x^3-\frac{3}{2}x^2+5x-\frac{1}{6}$$

22. Troba la primitiva de la funció *f*(*x*) = $x\sqrt{x^2-1}$ que s'anul·la quan *x* = 2.

$$F(x)=\int x\cdot\sqrt{x^2-1}\,dx=\frac{1}{2}\int 2x\cdot(x^2-1)^{1/2}dx=$$
$$=\frac{1}{2}\frac{(x^2-1)^{3/2}}{\frac{3}{2}}+C=\frac{1}{3}\sqrt{(x^2-1)^3}+C$$

$F(2) = 0 \rightarrow \frac{1}{3}\sqrt{27} + C = 0 \rightarrow C = \frac{1}{3}\cdot 3\sqrt{3} = -\sqrt{3}$

$F(x) = \frac{1}{3}\sqrt{(x^2-1)^3} - \sqrt{3}$

23. Calcula ∫tg² *x* d*x*.

Et suggerim que apliquis l'estratègia següent:

tg² *x* = 1 + tg² *x* – 1

$$\int tg^2x\,dx=\int(tg^2x+1-1)dx=$$
$$=\int(tg^2x+1)dx-\int dx=tgx+C$$

24. Calcula:

a) **∫5cos (3x – 2) *dx***

$$\int 5\cos(3x-2)dx=\frac{5}{3}\int 3\cos(3x-2)dx=$$
$$=\frac{5}{3}\sin(3x-2)+C$$

b) $\int \frac{1}{5x-12}\ dx$

$$\int \frac{1x}{5x-12}dx = \frac{1}{5}\int \frac{5}{5x-12}dx =$$
$$= \frac{1}{5}\ln|5x-12| + C$$

c) $\int \frac{7\sin\sqrt{x}}{3\sqrt{x}}\ dx$

$$\int \frac{7\sin\sqrt{x}}{3\sqrt{x}} = \frac{7\cdot 2}{3}\int \frac{\sin\sqrt{x}}{2\sqrt{x}}dx =$$
$$= \frac{-14}{3}\cos\sqrt{x} + C$$

d) $\int \frac{e^{x+1}}{e^x-5}\ dx$

$$\int \frac{e^{x+1}}{e^x-5}dx = \int \frac{e\cdot e^x}{e^x-5}dx = e\int \frac{e^x}{e^x-5}dx =$$
$$= e\cdot\ln|e^x-5| + C$$

e) $\int \sqrt{7x-6}\ dx$

$$\int \sqrt{7x-6}\,dx = \frac{1}{7}\int 7\cdot(7x-6)^{1/2}dx =$$
$$\frac{1}{7}\frac{(7x-6)^{3/2}}{\frac{3}{2}} + C = \frac{2}{21}\sqrt{(7x-6)^3} + C$$

f) $\int \frac{3}{\sqrt{5x+8}}\ dx$

$$\int \frac{3}{\sqrt{5x+8}}dx = \frac{3}{5}\int 5\cdot(5x+8)^{-1/2}dx =$$
$$= \frac{3}{5}\cdot\frac{(5x+8)^{1/2}}{\frac{1}{2}} + C = \frac{6}{5}\sqrt{5x+8} + C$$

g) $\int \frac{7}{1+4x^2}\ dx$

$$\int \frac{7}{1+4x^2}dx = \frac{7}{2}\int \frac{2}{1+(2x)^2}dx =$$
$$= \frac{7}{2}\text{arctg}2x + C$$

h) $\int \frac{5}{\sqrt{1-9x^2}}\ dx$

$$\int \frac{5}{\sqrt{1-9x^2}}dx = \frac{5}{3}\int \frac{3}{\sqrt{1-(3x)^2}}dx =$$
$$= \frac{5}{3}\arcsin 3x + C$$

i) $\int 2x^2\sqrt{1-x^3}\ dx$

$$\int 2x^2\sqrt{1-x^3}dx = -\frac{2}{3}\int -3x^2(1-x^3)^{1/2}dx =$$
$$= -\frac{2}{3}\cdot\frac{(1-x^3)^{3/2}}{\frac{3}{2}}dx + C = -\frac{4}{9}\sqrt{(1-x^3)^3} + C$$

j) $\int \frac{3}{4+100x^2}\ dx$

$$\int \frac{3}{4+100x^2}dx = \frac{3}{4}\int \frac{1}{1+25x^2}dx =$$
$$= \frac{3}{4}\cdot\frac{1}{5}\int \frac{5}{1+(5x)^2}dx = \frac{3}{20}arctg\,5x + C$$

25. Calcula:

a) $\int x \sin x\ dx$

$$f(x) = x \to f'(x) = 1$$
$$g'(x) = \sin x \to g(x) = -\cos x$$
$$\int x\sin x dx = -x\cos x + \int \cos x dx =$$
$$= -x\cos x + \sin x + C$$

b) $\int e^{2x} \sin x\ dx$

$$\int e^{2x}\cdot\sin x dx$$
$$f(x) = e^{2x} \to f'(x) = 2e^{2x}$$
$$g'(x) = \sin x \to g(x) = -\cos x$$
$$\int e^{2x}\sin x dx = -e^{2x}\cos x + 2\int e^{2x}\cos x dx$$
$$\int e^{2x}\cos x dx = e^{2x}\sin x - 2\int e^{2x}\sin x dx$$
$$r(x) = e^{2x} \to r'(x) = 2e^{2x}$$
$$s'(x) = \cos x \to s(x) = \sin x$$
$$\int e^{2x}\sin x dx = -e^{2x}\cos x +$$
$$+2(e^{2x}\sin x\,dx - 2\int e^{2x}\sin x\,dx)$$
$$\int e^{2x}\sin x dx = -e^{2x}\cos x + 2e^{2x}\sin x -$$
$$-4\int e^{2x}\sin x dx$$
$$\int e^{2x}\sin x dx + 4\int e^{2x}\sin x dx =$$
$$= e^{2x}(-\cos x + 2\sin x)$$
$$5\int e^{2x}\sin x dx = e^{2x}(-\cos x + 2\sin x)$$
$$\int e^{2x}\sin x dx = \frac{e^{2x}(-\cos x + 2\sin x)}{5} + C$$

c) **∫ln *x* d*x***

$$f(x) = \ln x \rightarrow f'(x) = 1/x$$
$$g'(x) = 1 \rightarrow g(x) = x$$
$$\int \ln x dx = x \ln x - \int \frac{1}{x} \cdot x dx = x \ln x - \int dx =$$
$$= x \ln x - x + C = x(\ln x - 1) + C$$

d) **∫*x* ln *x* d*x***

$$f(x) = \ln x \rightarrow f'(x) = 1/x$$
$$g'(x) = x \rightarrow g(x) = x^2/2$$
$$\int x \ln x dx = \frac{x^2}{2} \ln x - \int \frac{1}{x} \cdot \frac{x^2}{2} dx = \frac{x^2}{2} \ln x -$$
$$-\frac{1}{2} \int x dx = \frac{x^2}{2} \ln x - \frac{1}{2} \frac{x^2}{2} + C =$$
$$= \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right) + C$$

e) **∫2*ˣ x* d*x***

$$f(x) = x \rightarrow f'(x) = 1$$
$$g'(x) = 2^x \rightarrow g(x) = 2^x/\ln 2$$
$$\int 2^x x dx = x \frac{2^x}{\ln 2} - \frac{1}{\ln 2} - \int 2^x dx =$$
$$= \frac{x \cdot 2^x}{\ln 2} - \frac{1}{\ln 2} \cdot \frac{2^x}{\ln 2} + C = \frac{2^x}{\ln 2}\left(x - \frac{1}{\ln 2}\right) + C$$

f) **∫arc sin *x* d*x***

$$\int \arcsin x dx$$
$$f(x) = \arcsin x \rightarrow f'(x) = \frac{1}{\sqrt{1-x^2}}$$
$$g'(x) = 1 \rightarrow g(x) = x$$
$$\int \arcsin x dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} dx =$$
$$= x \arcsin x - \int x \cdot (1-x^2)^{-1/2} dx = x \arcsin x +$$
$$+\frac{1}{2} \int -2x(1-x^2)^{-1/2} dx = x \arcsin x +$$
$$+\frac{1}{2} \cdot \frac{(1-x^2)^{1/2}}{1/2} + C = x \arcsin x + \sqrt{1-x^2} + C$$

g) **∫(*x* + 2) e³ˣ d*x***

$$f(x) = x + 2 \rightarrow f'(x) = 1$$
$$g'(x) = e^{3x} \rightarrow g(x) = \frac{1}{3} e^{3x}$$
$$\int (x+2) e^{3x} dx = \frac{1}{3}(x+2)e^{3x} - \int \frac{1}{3} e^{3x} dx =$$
$$= \frac{e^{3x}}{3}(x+2) - \frac{1}{3} \cdot \frac{e^{3x}}{3} + C =$$
$$= \frac{e^{3x}}{3}\left(x + 2 - \frac{1}{3}\right) + C = \frac{e^{3x}}{3}\left(x + \frac{5}{3}\right) + C$$

h) $\int \frac{x}{3^x}$ **d*x***

$$\int \frac{x}{3^x} dx = \int x \cdot 3^{-x} dx$$
$$f(x) = x \rightarrow f'(x) = 1$$
$$g'(x) = 3^{-x} \rightarrow g(x) = \frac{-1}{\ln 3} 3^{-x}$$
$$\int x \cdot 3^{-x} dx = \frac{-x}{\ln 3} 3^{-x} + \frac{1}{\ln 3} \int 3^{-x} dx =$$
$$\frac{-x}{\ln 3} 3^{-x} - \frac{1}{\ln 3} \cdot \frac{1}{\ln 3} 3^{-x} + C =$$
$$= \frac{-1}{3^x \ln 3}\left(x + \frac{1}{\ln 3}\right) + C$$

i) **∫(3*x* + 2) cos *x* d*x***

$$\int (3x+2) \cos x dx$$
$$f(x) = 3x + 2 \rightarrow f'(x) = 3$$
$$g'(x) = \cos x \rightarrow g(x) = \sin x$$
$$\int (3x+2) \cos x dx = (3x+2) \sin x -$$
$$-3 \int \sin x dx = (3x+2) \sin x + 3 \cos x + C$$

j) $\int \frac{x}{e^{2x}}$ **d*x***

$$\int \frac{x}{e^{2x}} dx = \int x \cdot e^{-2x} dx$$
$$f(x) = x \rightarrow f'(x) = 1$$
$$g'(x) = e^{-2x} \rightarrow g(x) = -\frac{1}{2} e^{-2x}$$
$$\int x e^{-2x} dx = -\frac{1}{2} e^{-2x} \cdot x + \frac{1}{2} \int e^{-2x} dx =$$
$$= -\frac{1}{2} e^{-2x} \cdot x - \frac{1}{2} \cdot \frac{1}{2} e^{-2x} + C =$$
$$= -\frac{1}{2} e^{-2x}\left(x + \frac{1}{2}\right) + C = \frac{-1}{2e^{2x}}\left(x + \frac{1}{2}\right) + C$$

26. Ja has vist que, de vegades, cal aplicar en més d'una ocasió el mètode d'integració per parts. Et caldrà fer-ho en el càlcul de les primitives següents:

a) **∫*x*² e⁵ˣ d*x***

$$\int x^2 e^{5x} dx$$

$$f(x) = x^2 \to f'(x) = 2x$$

$$g'(x) = e^{5x} \to g(x) = \frac{1}{5} e^{5x}$$

$$\int x^2 e^{5x} dx = \frac{1}{5} x^2 e^{5x} - \frac{2}{5} \int x e^{5x} dx$$

$$\int x e^{5x} dx = \frac{1}{5} x e^{5x} - \frac{1}{5} \int e^{5x} dx = \frac{1}{5} x e^{5x} - \frac{1}{25} e^{5x}$$

$$r(x) = x \to r'(x) = 1$$

$$s'(x) = e^{5x} \to s(x) = \frac{1}{5} e^{5x}$$

$$\int x^2 e^{5x} dx = \frac{1}{5} x^2 e^{5x} - \frac{2}{5}\left(\frac{1}{5} x e^{5x} - \frac{1}{25} e^{5x}\right) + C$$

$$= \frac{1}{5} e^{5x}\left(x^2 - \frac{2}{5} x + \frac{2}{25}\right) + C$$

b) **$\int x^3 \sin x \, dx$**

$$\int x^3 \sin x dx$$

$$f(x) = x^3 \to f'(x) = 3x^2$$

$$g'(x) = \sin x \to g(x) = -\cos x$$

$$\int x^3 \sin x dx = -x^3 \cos x + 3 \int x^2 \cos x dx$$

$$\int x^2 \cos x dx = x^2 \sin x - 2 \int x \sin x dx$$

$$r(x) = x^2 \to r'(x) = 2x$$

$$s'(x) = \cos x \to s(x) = \sin x$$

$$\int x^3 \sin x dx = -x^3 \cos x +$$

$$+ 3(x^2 \sin x - 2 \int x \sin x dx) = -x^3 \cos x +$$

$$+3x^2 \sin x - 6 \int x \sin x dx$$

$$\int x \sin x dx = -x \cos x + \int \cos x dx =$$

$$= -x \cos x + \sin x$$

$$t(x) = x \to t'(x) = 1$$

$$n'(x) = \sin x \to n(x) = -\cos x$$

$$\int x^3 \sin x dx = -x^3 \cos x + 3x^2 \sin x +$$

$$+6x \cos x - 6 \sin x + C = (-x^3 + 6x) \cos x +$$

$$+(3x^2 - 6) \sin x + C$$

c) **$\int (x^2 + 4) \cdot 3^x \, dx$**

$$\int (x^2 + 4) \cdot 3^x dx$$

$$f(x) = x^2 + 4 \to f'(x) = 2x$$

$$g'(x) = 3^x \to g(x) = \frac{3^x}{\ln 3}$$

$$\int (x^2 + 4) \cdot 3^x dx = (x^2 + 4) \frac{3^x}{\ln 3} - \frac{2}{\ln 3} \int x \cdot 3^x dx$$

$$\int x \cdot 3^x dx = x \cdot \frac{3^x}{\ln 3} - \frac{1}{\ln 3} \int 3^x dx = \frac{x \cdot 3^x}{\ln 3} -$$

$$- \frac{1}{\ln 3} \cdot \frac{3^x}{\ln 3}$$

$$r(x) = x \to r'(x) = 1$$

$$s'(x) = 3^x \to s(x) = \frac{3^x}{\ln 3}$$

$$\int (x^2 + 4) 3^x dx = \frac{(x^2 + 4) 3^x}{\ln 3} -$$

$$- \frac{2}{\ln 3}\left(\frac{x \cdot 3^x}{\ln 3} - \frac{3^x}{(\ln 3)^2}\right) + C =$$

$$= \frac{3^x}{\ln 3}\left[x^2 - \frac{2}{\ln 3} x + 4 + \frac{2}{(\ln 3)^2}\right] + C$$

d) **$\int x^2 \cos x \, dx$**

$$\int x^2 \cos x dx$$

$$f(x) = x^2 \to f'(x) = 2x$$

$$g'(x) = \cos x \to g(x) = \sin x$$

$$\int x^2 \cos x dx = x^2 \sin x - 2 \int x \sin x dx$$

$$\int x \sin x dx = -x \cos x + \int \cos x dx =$$

$$= -x \cos x + \sin x$$

$$r(x) = x \to r'(x) = 1$$

$$s'(x) = \sin x \to s(x) = -\cos x$$

$$\int x^2 \cos x dx = x^2 \sin x - 2(-x \cos x + \sin x) +$$

$$+C = x^2 \sin x + 2x \cos x - 2 \sin x + C$$

e) **$\int \frac{x^2}{2^x} \, dx$**

$$\int \frac{x^2}{2^x} dx = \int x^2 2^{-x} dx$$

$$f(x) = x^2 \to f'(x) = 2x$$

$$g'(x) = 2^{-x} \to g(x) = \frac{-2^{-x}}{\ln 2}$$

$$\int x^2 2^{-x} dx = \frac{-x^2 \cdot 2^{-x}}{\ln 2} + \frac{2}{\ln 2} \int x \cdot 2^{-x} dx$$

$$\int x \cdot 2^{-x} dx = \frac{-x 2^{-x}}{\ln 2} + \int \frac{2^{-x}}{\ln 2} dx =$$

$$= \frac{-x 2^{-x}}{\ln 2} - \frac{1}{\ln 2} \frac{2^{-x}}{\ln 2}$$

$$r(x) = x \to r'(x) = 1$$

$$s'(x) = 2^{-x} \to s(x) = \frac{-2^{-x}}{\ln 2}$$

$$\int x^2 2^{-x}\,dx = \frac{-x^2 \cdot 2^{-x}}{\ln 2} +$$

$$+\frac{2}{\ln 2}\left(\frac{-x \cdot 2^{-x}}{\ln 2} - \frac{1}{\ln 2}\frac{2^{-x}}{\ln 2}\right) + C =$$

$$= \frac{-1}{2^x \ln 2}\left[x^2 + \frac{2}{\ln 2}x + \frac{2}{(\ln 2)^2}\right] + C$$

***f*)** $\int (1 - x^2)\, 2^{3x}\, dx$

$$\int (1 - x^2) 2^{3x}\, dx$$

$$f(x) = 1 - x^2 \to f'(x) = -2x$$

$$g'(x) = 2^{3x} \to g(x) = \frac{1}{3\ln 2} \cdot 2^{3x}$$

$$\int (1 - x^2) 2^{3x}\,dx = (1 - x^2) \cdot \frac{1}{3\ln 2} 2^{3x} +$$

$$+\frac{2}{3\ln 2}\int x \cdot 2^{3x}\,dx$$

$$\int x 2^{3x}\,dx = x \cdot \frac{1}{3\ln 2} 2^{3x} - \frac{1}{3\ln 2}\int 2^{3x} dx =$$

$$= \frac{x \cdot 2^{3x}}{3\ln 2} - \frac{2^{3x}}{(3\ln 2)^2}$$

$$r(x) = x \to r'(x) = 1$$

$$s'(x) = 2^{3x} \to s(x) = \frac{1}{3\ln 2} 2^{3x}$$

$$\int (1 - x^2) 2^{3x}\,dx = \frac{(1 - x^2) 2^{3x}}{3\ln 2} +$$

$$+\frac{2}{3\ln 2}\left[\frac{x 2^{3x}}{3\ln 2} - \frac{2^{3x}}{(3\ln 2)^2}\right] + C =$$

$$= \frac{2^{3x}}{3\ln 2}\left[-x^2 + \frac{2x}{3\ln 2} - \frac{2}{(3\ln 2)^2} + 1\right] + C$$

27. *a*) Resol l'equació *F'*(*x*) = 0 si *F*(*x*) = ∫e*ˣ* *x* (2 + *x*) d*x*.

$$F(x) = \int e^x x(x + 2)dx$$

$$F'(x) = e^x x(x + 2)$$

$$F'(x) = 0 \to e^x x(x + 2) = 0 \left\langle \begin{array}{l} x = 0 \\ x + 2 = 0 \to x = -2 \end{array} \right.$$

Les solucions són $x_1 = 0$ i $x_2 = -2$

***b*) Calcula la primitiva de la funció *f*(*x*) = e*ˣ* *x* (2 + *x*) la gràfica de la qual passa per l'origen de coordenades.**

$$\int e^x (x^2 + 2x)dx =$$

$$f(x) = x^2 + 2x \to f'(x) = 2x + 2 = 2(x + 1)$$

$$g'(x) = e^x \to g(x) = e^x$$

$$\int e^x (x^2 + 2x)dx = (x^2 + 2x)e^x - 2\int (x + 1)e^x dx$$

$$\int (x + 1)e^x dx = (x + 1)e^x - \int e^x dx =$$

$$= (x + 1)e^x - e^x$$

$$r(x) = x + 1 \to r'(x) = 1$$

$$s'(x) = e^x \to s(x) = e^x$$

$$F(x) = \int e^x (x^2 + 2x)dx = (x^2 + 2x)e^x -$$

$$-2[(x + 1)e^x - e^x] + C = x^2 \cdot e^x + 2xe^x -$$

$$-2xe^x - 2e^x + C = x^2 e^x + C$$

$$F(0) = 0 \to 0 = 0 + C \to C = 0 \to$$

$$\to F(x) = x^2 e^x$$

28. Calcula amb els canvis de variable indicats:

a*)** $\int \frac{1}{\sqrt{16 - x^2}}$ **d*x* amb *x* = 4*t

$$\int \frac{1}{\sqrt{16 - x^2}} dx; x = 4t \to dx = 4dt$$

$$\int \frac{1}{\sqrt{16 - x^2}} dx = \int \frac{1}{\sqrt{16 - 16t^2}} 4dt = \int \frac{1}{\sqrt{1 - t^2}} dt$$

$$= \arcsin t + C = \arcsin \frac{x}{4} + C$$

***b*)** $\int \frac{x}{\sqrt{x - 1}}$ **d*x* amb** $\sqrt{x - 1} = t$

$$\sqrt{x - 1} = t \to x - 1 = t^2 \to x^2 = t^2 + 1 \to$$

$$\to dx = 2 + dt$$

$$\int \frac{x}{\sqrt{x - 1}} dx = \int \frac{t^2 + 1}{t} 2tdt = 2\int (t^2 + 1)dt =$$

$$= 2\left(\frac{t^3}{3} + t\right) + C = 2\left(\frac{\sqrt{(x - 1)^3}}{3} + \sqrt{x - 1}\right) + C =$$

$$= 2\sqrt{x - 1}\left(\frac{x - 1}{3} + 1\right) = \frac{2}{3}\sqrt{x - 1}(x + 2)$$

29. Aplicant el canvi de variable sin *x* = *t*, calcula:

$$\int \cos^3 x\, dx$$

Si tens en compte la igualtat següent:

$$\cos^3 x = \cos^2 x \cos x = (1 - \sin^2 x)\cos x$$

la pots calcular sense canvi de variable. Fes-ho.

$$\sin x = t \rightarrow dt = \cos x dx \rightarrow dx = \frac{dt}{\cos x}$$

$$\int \cos^3 x dx = \int \cos^3 x \frac{dt}{\cos x} = \int \cos^2 x dt =$$

$$= \int (1-\sin^2 x) dt = \int (1-t^2) dt = t - \frac{t^3}{3} + C =$$

$$= \sin x - \frac{\sin^3 x}{3} + C = \sin x\left(1 - \frac{\sin^2 x}{3}\right) + C$$

$$\int \cos^3 x dx = \int \cos^2 x \cos x dx =$$

$$= \int (1-\sin^2 x)\cos x dx = \int (\cos x - \sin^2 x \cos x) dx =$$

$$\sin x - \frac{\sin^3 x}{3} + C = \sin x\left(1 - \frac{\sin^2 x}{3}\right) + C$$

30. Calcula la integral $\int \sqrt{1-x^2}\ dx$ utilitzant el canvi de variable *x* = sin *t* o *x* = cos *t*. Arribaràs a una integral del tipus:

$$\int \cos^2 t\, dt \text{ o } \int \sin^2 t\, dt$$

respectivament. Et caldrà fer ús de les identitats trigonomètriques:

$$\cos^2 t = \frac{1+\cos 2t}{2} \quad \text{o} \quad \sin^2 t = \frac{1-\cos 2t}{2}$$

$$x = \sin t \rightarrow dx = \cos t dt$$

$$\int \sqrt{1-x^2} dx = \int \sqrt{1-\sin^2 t} \cos t dt = \int \cos^2 t dt =$$

$$= \int \frac{1+\cos 2t}{2} dt = \int \left(\frac{1}{2} + \frac{\cos 2t}{2}\right) dt = \frac{t}{2} + \frac{\sin 2t}{4} +$$

$$+C = \frac{t}{2} + \frac{2\sin t \cos t}{4} + C = \frac{t}{2} + \frac{\sin t \cos t}{2} + C =$$

$$= \frac{1}{2}(t + \sin t \cos t) + C = \frac{1}{2}(\arcsin x + x\sqrt{1-x^2}) +$$

$$+C$$

$$x = \cos t \rightarrow dx = -\sin t dt$$

$$\int \sqrt{1-x^2} dx = -\int \sqrt{1-\cos^2 t} \sin t dt = -\int \sin^2 t dt =$$

$$= -\int \frac{1-\cos 2t}{2} dt = -\left(\frac{t}{2} - \frac{\sin 2t}{4}\right) + C =$$

$$= -\left(\frac{t}{2} - \frac{2\sin t \cos t}{4}\right) + C = \frac{1}{2}(-t + \sin t \cos t) + C$$

$$= -\frac{1}{2}(-\arccos x + x\sqrt{1-x^2}) + C$$

31. La integral $\int \frac{x}{x^2+3}$ d*x* és quasi immediata.

Calcula-la.

Comprova que arribes al mateix resultat aplicant-hi el canvi de variable $x^2 + 3 = t$.

$$\int \frac{x}{x^2+3} dx = \frac{1}{2}\int \frac{2x}{x^2+3} dx = \frac{1}{2}\ln(x^2+3) + C =$$

$$= \ln\sqrt{x^2+3} + C$$

$$x^2 + 3 = t \rightarrow dt = 2x dx$$

$$\int \frac{x}{x^2+3} dx = \frac{1}{2}\int \frac{2x}{x^2+3} dx = \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\ln|t| + C =$$

$$= \frac{1}{2}\ln|x^2+3| + C = \frac{1}{2}\ln(x^2+3) + C =$$

$$= \ln\sqrt{x^2+3} + C$$

32. Calcula:

a) $\displaystyle\int \frac{10}{x^2-4x+4} dx$

$$\int \frac{10}{x^2-4x+4} dx = \int \frac{10}{(x-2)^2} dx =$$

$$= 10\int (x-2)^{-2} dx = 10 \cdot \frac{(x-2)^{-1}}{-1} + C =$$

$$= \frac{-10}{x-2} + C$$

b) $\displaystyle\int \frac{3x-2}{x^2-9} dx$

$$\frac{3x-2}{x^2-9} = \frac{A}{x+3} + \frac{B}{x-3}$$

$$3x - 2 = A(x-3) + B(x+3)$$

$$x = 3 \rightarrow 7 = B \cdot 6 \rightarrow B = 7/6$$

$$x = -3 \rightarrow -11 = A \cdot (-6) \rightarrow A = 11/6$$

$$\int \frac{3x-2}{x^2-9} dx = \int \frac{11/6}{x+3} dx + \int \frac{7/6}{x-3} dx =$$

$$= \frac{11}{6}\ln|x+3| + \frac{7}{6}\ln|x-3| + C$$

c) $\displaystyle\int \frac{\boldsymbol{x^2 - 3}}{\boldsymbol{x(x-1)(x+2)}}\,\boldsymbol{dx}$

$$\frac{x^2-3}{x(x-1)(x+2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2}$$

$$x^2 - 3 = A(x-1)(x+2) + Bx(x+2) +$$

$$+Cx(x-1)$$

$$x = 0 \to -3 = A\cdot(-2) \to A = 3/2$$
$$x = 1 \to -2 = B\cdot 3 \to A = -2/3$$
$$x = -2 \to 1 = C\cdot 6 \to C = 1/6$$

$$\int \frac{x^2-3}{x(x-1)(x+2)}dx = \int \frac{3/2}{x}dx + \int \frac{-2/3}{x-1}dx +$$

$$+\int \frac{1/6}{x+2}dx = \frac{3}{2}\ln|x| - \frac{2}{3}\ln|x-1| +$$

$$+\frac{1}{6}\ln|x+2| + C$$

d) $\displaystyle\int \frac{\boldsymbol{3}}{\boldsymbol{x^2 - x}}\,\boldsymbol{dx}$

$$\frac{3}{x^2-x} = \frac{3}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$$

$$3 = A(x-1) + Bx$$
$$x = 0 \to 3 = A\cdot(-1) \to A = -3$$
$$x = 1 \to 3 = B \to B = 3$$

$$\int \frac{3}{x(x-1)}dx = \int \frac{-3}{x}dx + \int \frac{3}{x-1}dx = -3\ln|x| +$$

$$+3\ln|x-1| + C = 3\ln\left|\frac{x-1}{x}\right| + C = \ln\left|\frac{x-1}{x}\right|^3 + C$$

e) $\displaystyle\int \frac{\boldsymbol{1-2x}}{\boldsymbol{x^3 - 6x^2 + 11x - 6}}\,\boldsymbol{dx}$

$$x^3 - 6x^2 + 11x - 6 = 0 \to x_1 = 1, x_2 = 2, x_3 = 3$$

$$x^3 - 6x^2 + 11x - 6 = (x-1)(x-2)(x-3)$$

$$\frac{1-2x}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

$$1 - 2x = A(x-2)(x-3) + B(x-1)(x-3) +$$

$$+C(x-1)(x-2)$$

$$x = 1 \to -1 = A\cdot 2 \to A = -1/2$$
$$x = 2 \to -3 = B\cdot(-1) \to B = 3$$
$$x = 3 \to -5 = C\cdot 2 \to C = -5/2$$

$$\int \frac{1-2x}{(x-1)(x-2)(x-3)}dx = \int \frac{-1/2}{x-1}dx +$$

$$+\int \frac{3}{x-2}dx + \int \frac{-5/2}{x-3}dx = -\frac{1}{2}\ln|x-1| +$$

$$+3\ln|x-2| - \frac{5}{2}\ln|x-3| + C$$

f) $\displaystyle\int \frac{\boldsymbol{x-3}}{\boldsymbol{x^2 - 6x + 5}}\,\boldsymbol{dx}$

$$\int \frac{x-3}{x^2-6x+5}dx = \frac{1}{2}\int \frac{2x-6}{x^2-6x+5}dx =$$
$$= \frac{1}{2}\ln|x^2 - 6x + 5| + C$$

33. Calcula $\displaystyle\int \frac{\boldsymbol{1}}{\boldsymbol{\sqrt[3]{x} + \sqrt{x}}}\,\boldsymbol{dx}$ **fent el canvi de variable** $\boldsymbol{x = t^6}$**.**

$$x = t^6 \to dx = 6t^5 dt$$
$$\searrow \sqrt[3]{x} = \sqrt[3]{t^6} = t^2; \sqrt{x} = \sqrt{t^6} = t^3$$

$$\int \frac{1}{\sqrt[3]{x} + \sqrt{x}}dx = \int \frac{1}{t^2 + t^3}6t^5 dt = 6\int \frac{t^5}{t^2 + t^3}dt =$$

$$= 6\int \frac{t^3}{t+1}dt = 6\int\left(t^2 - t + 1 - \frac{1}{t+1}\right)dt =$$

$$= 6\left(\frac{t^3}{3} - \frac{t^2}{2} + t - \ln|t+1|\right) + C =$$

$$= 2t^3 - 3t^2 + 6t - 6\ln|t+1| + C =$$

$$= 2\sqrt{2} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6\ln|\sqrt[6]{x} + 1| + C$$

$$\begin{array}{l|l} t^3 & t^2 + 1 \\ \hline -t^3 - t^2 & t^2 - t + 1 \\ \quad -t^2 & \\ \quad +t^2 + t & \\ \qquad t & \\ \qquad -t + 1 & \\ \qquad\quad -1 & \end{array}$$

34. Calcula:

a) $\displaystyle\int \frac{\boldsymbol{x^2}}{\boldsymbol{x-5}}\,\boldsymbol{dx}$

$$\int \frac{x^2}{x-5}dx = \int\left(x + 5 + \frac{25}{x-5}\right)dx =$$

$$= \frac{x^2}{2} + 5x + 25\ln|x-5| + C$$

$$\begin{array}{l|l} x^2 & x-5 \\ \hline -x^2 + 5x & x+5 \\ \quad 5x & \\ \quad -5x + 25 & \\ \qquad 25 & \end{array}$$

b) $\displaystyle\int \frac{x^3-4}{x^2-2x}dx$

$$\int \frac{x^3-4}{x^2-2x}dx = \int\left(x+2+\frac{4x-4}{x^2-2x}\right)dx =$$

$$= \int (x+2)dx + 2\int\frac{2x-2}{x^2-2x}dx =$$

$$= \frac{x^2}{2} + 2x + 2\ln|x^2-2x| + C$$

$$\begin{array}{l|l} x^3 - 4 & x^2 - 2x \\ \hline -x^3 - 2x^2 & x+2 \\ \quad 2x^2 - 4 & \\ \quad -2x^2 + 4x & \\ \qquad 4x - 4 & \end{array}$$

c) $\displaystyle\int \frac{1}{x^2(x-1)^2}dx$

$$\frac{1}{x^2(x-1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$

$$\frac{1}{x^2(x-1)^2} =$$

$$= \frac{Ax(x-1)^2 + Bx(x-1)^2 + Cx^2(x-1) + Dx^2}{x^2(x-1)^2}$$

$$1 = Ax(x-1)^2 + B(x-1)^2 + Cx^2(x-1) + Dx^2$$

$x = 0 \to 1 = B \to B = 1$

$x = 1 \to 1 = D \to D = 1$

$x = 2 \to 1 = A\cdot 2 + B + C\cdot 4 + D\cdot 4 \to A + 2C = -2$

$x = -1 \to 1 = A\cdot(-4) + B\cdot 4 + C\cdot(-2) + D \to -2A - C = -2$

$A = 2, C = -2$

$$\int\frac{1}{x^2(x-1)^2}dx = \int\frac{2}{x}dx + \int x^{-2}dx + \int\frac{-2}{x-1}dx +$$

$$+\int (x-1)^{-2}dx = 2\ln|x| - 2\ln|x-1| + \frac{x^{-1}}{-1} +$$

$$+\frac{(x-1)^{-1}}{-1} + C = 2\ln\left|\frac{x}{x-1}\right| - \frac{1}{x} - \frac{1}{x-1} + C$$

d) $\displaystyle\int \frac{x^2+9}{x^2-9}dx$

$$\int\frac{x^2+9}{x^2-9}dx = \int\left(1+\frac{18}{x^2-9}\right)dx = x +$$

$$+18\int\frac{1}{x^2-9}dx$$

$$\frac{1}{(x+3)(x-3)} = \frac{A}{x+3} + \frac{B}{x-3}$$

$1 = A(x-3) + B(x+3)$

$x = 3 \to 1 = B\cdot 6 \to B = 1/6$

$x = -3 \to 1 = A\cdot(-6) \to A = -1/6$

$$\int\frac{x^2+9}{x^2-9}dx = x + 18\left(\int\frac{-1/6}{x+3}dx + \int\frac{1/6}{x-3}dx\right) =$$

$$= x + 3(\ln|x-3| - \ln|x+3|) + C =$$

$$= x + \ln\left|\frac{x-3}{x+3}\right|^3 + C$$

$$\begin{array}{l|l} x^2 + 9 & x^2 - 9 \\ \hline -x^2 + 9 & 1 \\ \quad 18 & \end{array}$$

Acabem

1. **Determina la funció $f(x)$ sabent que la funció $F(x) = x^2 e^x + 2$ n'és una primitiva.**

$$f(x) = F'(x) = 2x\cdot e^x + x^2\cdot e^x = xe^x(2+x)$$

2. **Quina és la primitiva de la funció $f(x) = 2x + 5$ que verifica la condició $F(1) = 9$? I la que verifica $F^{-1}(3) = -3$?**

$F(x) = \int (2x+5)dx = x^2 + 5x + C$

$F_1(1) = 9 \to 9 = 1 + 5 + C \to C = 3 \to$

$\to F_1(x) = x^2 + 5x + 3$

$F_2^{-1}(3) = -3 \to 3 = 9 - 15 + C \to C = 9 \to$

$\to F_2(x) = x^2 + 5x + 9$

3. **Dos companys obtenen resultats diferents en el càlcul de les primitives d'una mateixa funció. El primer obté:**

$\int$**cos² 3*x* d*x*** $= \dfrac{x}{2} - \dfrac{\sin 6x}{12} + C$ **i el segon,**

$\int$**cos² 3*x* d*x*** $= \dfrac{x}{2} + \dfrac{\sin 6x}{12} + C$

Indica raonadament quin dels dos ha arribat a la resposta correcta.

El segon, ja que si $F(x)=\frac{x}{2}+\frac{\sin 6x}{12}+C'$, es compleix que $F'(x)=\frac{1}{2}+\frac{1}{2}\cos 6x=\frac{1}{2}+$

$+\frac{1}{2}\cos(3x+3x)=\frac{1}{2}+\frac{1}{2}(\cos^2 3x-\sin^2 3x)=$

$=\frac{1}{2}+\frac{1}{2}(2\cos^2 3x-1)=\frac{1}{2}+\cos^2 3x-\frac{1}{2}=\cos^2 3x$

4. **Troba l'expressió de la funció *F*(*x*) la gràfica de la qual passa pel punt (1, 1) sabent que el pendent de la recta tangent en qualsevol punt ve donat per la funció $m(x) = 3x^2 + 6x - 4$.**

$$F(x)=\int(3x^2+6x-4)dx=x^3+3x^2-4x+C$$

$$F(1)=1\to 1=1+3-4+C\to C=1$$
$$F(x)=x^3+3x^2-4x+1$$

5. **Considera la funció $F(x)=\int\frac{5}{(x-4)^2}dx$.**

Determina'n les asímptotes, sabent que $F(0) = -3$.

$$F(x)=\int\frac{5}{(x-4)^2}dx=5\int(x-4)^{-2}dx=\frac{-5}{x-4}+C$$

$$F(0)=-3\to -3=\frac{5}{4}+C\to C=-17/4$$
$$F(x)=\frac{-5}{x-4}-\frac{17}{4}=\frac{-20-17x+68}{4x-16}=\frac{-17x+48}{4x-16}$$

$$4x-16=0\to x=4$$

Asímptota vertical $x = 4$

$$\lim_{x\to\infty}F(x)=\frac{-17}{4}$$

Asímptota horitzonal $y=\frac{-17}{4}$

6. **Calcula:**

a) $\int(2x^3-3x^2-1)\,dx$

$$\int(2x^3-3x^2-1)dx=\frac{1}{2}x^4-x^3-x+C$$

b) $\int(-3\cdot 3^x+4\cos x)\,dx$

$$\int(-3\cdot 3^x+4\cos x)dx=\frac{-3^{x+1}}{\ln 3}+4\sin x+C$$

c) $\int\frac{x^2-3\sqrt{x}+1}{x}dx$

$$\int\frac{x^2-3\sqrt{x}+1}{x}dx=\int\left(x-3x^{-1/2}+\frac{1}{x}\right)dx=$$
$$=\frac{1}{2}x^2-6\sqrt{x}+\ln|x|+C$$

d) $\int\frac{1+2x+3x^2}{4x^2}dx$

$$\int\frac{1+2x+3x^2}{4x^2}dx=\int\left(\frac{1}{4}x^{-2}+\frac{1}{2}\cdot\frac{1}{x}+\frac{3}{4}\right)dx=$$
$$=-\frac{1}{4x}+\frac{1}{2}\ln|x|+\frac{3}{4}x+C$$

e) $\int(4+\text{tg}^2\,x)dx$

$$\int(4+\text{tg}^2x)dx=\int(3+1+\text{tg}^2x)dx=3x+$$
$$+\text{tg}x+C$$

f) $\int\frac{2-4x}{7x}dx$

$$\int\frac{2-4x}{7x}dx=\int\left(\frac{2}{7}\cdot\frac{1}{x}-\frac{4}{7}\right)dx=\frac{2}{7}\ln|x|-\frac{4}{7}x+C$$

7. **Calcula les integrals quasi immediates següents:**

a) $\int\cos 5x\,dx$

$$\int\cos 5x\,dx=\frac{1}{5}\int 5\cos 5x\,dx=\frac{1}{5}\sin 5x+C$$

b) $\int\frac{1}{\cos^2 x\;tgx}dx$

$$\int\frac{1}{\cos^2 x\,\text{tg}x}dx=\int\frac{1/\cos^2 x}{\text{tg}x}dx=\ln|\text{tg}x|+C$$

c) $\int 5\sin^4 x\cos x\,dx$

$$\int 5\sin^4 x\cos x\,dx=\sin^5 x+C$$

d) $\int\frac{2^{x+2}}{2^x-13}dx$

$$\int\frac{2^{x+2}}{2^x-13}dx=4\int\frac{2^x}{2^x-13}dx=\frac{4}{\ln 2}\int\frac{2^x\cdot\ln 2}{2^x-13}dx=$$
$$=\frac{4}{\ln 2}\ln\left|2^x-13\right|+C$$

e) $\int \frac{3x^2}{x+7}dx$

$$\int \frac{3x^2}{x+7}dx = \int\left(3x - 21 + \frac{147}{x+7}\right)dx =$$
$$= \frac{3}{2}x^2 - 21x + 147\ln|x+7| + C$$

$$\begin{array}{l|l} 3x^2 & x+7 \\ \hline -3x^2 - 21x & 3x - 21 \\ \quad -21x & \\ \quad 21x + 147 & \\ \quad\quad 147 & \end{array}$$

f) $\int x^3 \sin(x^4 - \pi)\, dx$

$$\int x^3 \cdot \sin(x^4 - \pi)dx = \frac{1}{4}\int 4x^3 \cdot \sin(x^4 - \pi) =$$
$$= -\frac{1}{4}\cdot\cos(x^4 - \pi)$$

8. Calcula les integrals quasi immediates següents:

a) $\int 5^{\mathrm{tg}\,x}(1 + \mathrm{tg}^2 x)\, dx$

$$\int 5^{\mathrm{tg}x}(1 + \mathrm{tg}^2 x)dx = \frac{5^{\mathrm{tg}x}}{\ln 5} + C$$

b) $\int \frac{x^2}{4\sqrt{1-x^2}}dx$

$$\int \frac{x^2}{4\sqrt{1-x^3}}dx = \frac{-1}{4}\cdot\frac{1}{3}\int -3x^2(1-x^3)^{-1/2}dx =$$
$$= -\frac{1}{12}\cdot\frac{(1-x^3)^{1/2}}{1/2} + C = -\frac{1}{6}\sqrt{1-x^3} + C$$

c) $\int \frac{7}{x^2 - 8x + 16}dx$

$$\int \frac{7}{x^2 - 8x + 16}dx = 7\int (x-4)^{-2}dx = \frac{-7}{x-4} + C$$

d) $\int \frac{5\cos\sqrt{x}}{\sqrt{x}}dx$

$$\int \frac{5\cos\sqrt{x}}{\sqrt{x}}dx = 10\int \cos\sqrt{x}\cdot\frac{1}{2\sqrt{x}}dx =$$
$$= 10\sin\sqrt{x} + C$$

e) $\int \frac{(1+\ln x)^2}{4x}dx$

$$\int \frac{(1+\ln x)^2}{4x}dx = \frac{1}{4}\int (1+\ln x)^2\cdot\frac{1}{x}dx =$$
$$= \frac{1}{12}(1+\ln x)^2 + C$$

f) $\int \frac{1}{x^2}\sin\frac{1}{x}dx$

$$\int \frac{1}{x^2}\sin\frac{1}{x}dx = \cos\frac{1}{x} + C$$

9. Troba la primitiva de la funció:

$$f(x) = \frac{e^{\mathrm{tg}\,x}}{1 - \sin^2 x}$$

que verifica la condició $F\left(\frac{\pi}{4}\right) = e$.

$$F(x) = \int \frac{e^{\mathrm{tg}x}}{1-\sin^2 x}dx = \int e^{\mathrm{tg}x}\cdot\frac{1}{\cos^2 x}dx = e^{\mathrm{tg}x} + C$$
$$F\left(\frac{\pi}{4}\right) = e \to e = e^{\mathrm{tg}\frac{\pi}{4}} + C \to e = e + C \to C = 0$$
$$F(x) = e^{\mathrm{tg}x}$$

10. Calcula per parts les integrals següents:

a) $\int x^2 \sin 3x\, dx$

$$\int x^2 \sin 3x dx$$
$$f(x) = x^2 \to f'(x) = 2x$$
$$g'(x) = \sin 3x \to g(x) = -\frac{1}{3}\cos 3x$$
$$\int x^2\sin 3x dx = -\frac{1}{3}x^2\cos 3x + \frac{2}{3}\int x\cos 3x dx$$
$$\int x\cos 3x dx = \frac{1}{3}x\sin 3x - \frac{1}{3}\int \sin 3x dx =$$
$$= \frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x$$
$$r(x) = x \to r'(x) = 1$$
$$s'(x) = \cos 3x \to s(x) = \frac{1}{3}\sin 3x$$
$$\int x^2\sin 3x dx = -\frac{1}{3}x^2\cos 3x + \frac{2}{9}x\sin 3x +$$
$$+ \frac{2}{27}\cos x + C =$$
$$= \frac{1}{3}\left(-x^2\cos 3x + \frac{2}{3}x\sin 3x + \frac{2}{9}\cos 3x\right) + C$$

b) $\int \cos(\ln x)\, dx$

$$\int \cos(\ln x)dx$$
$$f(x) = \cos(\ln x) \to f'(x) = -\sin(\ln x)\cdot\frac{1}{x}$$
$$g'(x) = 1 \to g(x) = x$$
$$\int \cos(\ln x)dx = x\cos(\ln x) + \int \sin(\ln x)dx$$
$$\int \sin(\ln x)dx$$

$$r(x) = \sin(\ln x) \rightarrow r'(x) = -\cos(\ln x) \cdot \frac{1}{x}$$
$$s'(x) = 1 \rightarrow s(x) = x$$
$$\int \sin(\ln x)dx = x\sin(\ln x) - \int \cos(\ln x)dx$$
$$\int \cos(\ln x)dx = x\cos(\ln x) + x\sin(\ln x) - $$
$$-\int \cos(\ln x)dx$$
$$2\int \cos(\ln x)dx = x\left[\cos(\ln x) + \sin(\ln x)\right]$$
$$\int \cos(\ln x)dx = \frac{x\left[\cos(\ln x) + \sin(\ln x)\right]}{2} + C$$

c) **∫*x*³ ln *x* d*x***

$$\int x^3 \ln x dx = \frac{x^4}{4}\ln x - \frac{1}{4}\int x^3 dx = \frac{x^4}{4}\ln x - $$
$$-\frac{1}{4}\cdot\frac{x^4}{4} + C = \frac{x^4}{4}\left(\ln x - \frac{1}{4}\right) + C$$
$$f(x) = (\ln x) \rightarrow f'(x) = \frac{1}{x}$$
$$g'(x) = x^3 \rightarrow g(x) = \frac{x^4}{4}$$

d) **∫e^4*x* cos 4*x* d*x***

$$\int e^{4x}\cos 4x dx$$
$$f(x) = e^{4x} \rightarrow f'(x) = 4e^{4x}$$
$$g'(x) = \cos 4x \rightarrow g(x) = \frac{1}{4}\sin 4x$$
$$\int e^{4x}\cos 4x dx = \frac{1}{4}e^{4x}\sin 4x - \int e^{4x}\sin 4x dx =$$
$$\int e^{4x}\sin 4x dx$$
$$r(x) = e^{4x} \rightarrow r'(x) = 4e^{4x}$$
$$s'(x) = \sin 4x \rightarrow s(x) = -\frac{1}{4}\cos 4x$$
$$\int e^{4x}\sin 4x dx = -\frac{1}{4}e^{4x}\cos 4x + \int e^{4x}\cos 4x dx$$
$$\int e^{4x}\cos 4x dx = \frac{1}{4}e^{4x}\sin 4x + \frac{1}{4}e^{4x}\cos 4x - $$
$$-\int e^{4x}\cos 4x dx$$
$$2\int e^{4x}\cos 4x dx = \frac{1}{4}e^{4x}(\sin 4x + \cos 4x)$$
$$\int e^{4x}\cos 4x dx = \frac{1}{8}e^{4x}(\sin 4x + \cos 4x) + C$$

e) **∫(*x* – 1) 5^*x* d*x***

$$\int (x-1)5^x dx$$
$$f(x) = x - 1 \rightarrow f'(x) = 1$$
$$g'(x) = 5^x \rightarrow g(x) = \frac{1}{\ln 5}5^x$$
$$\int (x-1)5^x dx = \frac{1}{\ln 5}(x-1)5^x - \frac{1}{\ln 5}\int 5^x dx =$$
$$= \frac{1}{\ln 5}(x-1)5^x - \frac{1}{\ln 5}\cdot\frac{1}{\ln 5}5^x =$$
$$= \frac{5^x}{\ln 5}\left(x - 1 - \frac{1}{\ln 5}\right) + C$$

f) **∫ln² *x* d*x***

$$\int \ln^2 x dx = x\ln^2 x - 2\int \ln x dx$$
$$f(x) = \ln^2 x \rightarrow f'(x) = 2\ln x\frac{1}{x}$$
$$g'(x) = 1 \rightarrow g(x) = x$$
$$\int \ln x dx = x\ln x - \int dx = x\ln x - x$$
$$r(x) = \ln x \rightarrow r'(x) = \frac{1}{x}$$
$$s'(x) = 1 \rightarrow s(x) = x$$
$$\int \ln^2 x dx = x\ln^2 x - 2(x\ln x - x) + C =$$
$$= x(\ln^2 x - 2\ln x + 2) + C$$

11. Comprova que:

∫(*x*² – 2*x* – 1) e^*x* d*x* = (*x*² – 4*x* + 3) e^*x* + *C*

$$\int (x^2 - 2x - 1)e^x dx = (x^2 - 4x + 3)e^x + C$$
$$\left[(x^2 - 4x + 3)e^x + C\right]' = (2x - 4)e^x +$$
$$+(x^2 - 4x + 3)\cdot e^x = (2x - 4 + x^2 - 4x + 3)e^x =$$
$$= (x^2 - 2x - 1)e^x$$

12. Calcula les integrals següents, fent ús en cada cas del canvi de variable indicat:

a) $\int \sqrt{4 - x^2}$ **d*x*,** **x = 2 sin t**

$$\int \sqrt{4 - x^2 dx}$$
$$x = 2\sin t \rightarrow dx = 2\cos t dt$$
$$\int \sqrt{4 - x^2}dx = \int \sqrt{4 - 4\sin^2 t}\cdot 2\cos t dt =$$
$$= 4\int \cos^2 t dt = 4\int \frac{1 + \cos 2t}{2}dt =$$
$$= 4\int\left(\frac{1}{2} + \frac{\cos 2t}{2}\right)dt = 4\left(\frac{t}{2} + \frac{\sin 2t}{4}\right) + C =$$
$$= 2t + \sin 2t + C = 2\arcsin\frac{x}{2} + \frac{x}{2}\sqrt{4 - x^2} + C$$

b) $\displaystyle\int \frac{3}{9+x^2}\,dx$, $x = 3t$

$$\int \frac{3}{9+x^2}dx$$

$$x = 3t \to dx = 3dt$$

$$\int \frac{3}{9+9t^2}\cdot 3dt = \int \frac{1}{1+t^2}dt = \operatorname{arctg} t + C =$$

$$= \operatorname{arctg}\frac{x}{3} + C$$

c) $\displaystyle\int \frac{\sqrt{x}}{x-1}\,dx$, $\sqrt{x} = t$

$$\int \frac{\sqrt{x}}{x-1}dx$$

$$\sqrt{x} = t \to dt = \frac{1}{2\sqrt{x}}dx \to dx = 2\sqrt{x}dt = 2tdt$$

$$\int \frac{t}{t^2-1}\cdot 2tdt = 2\int \frac{t^2}{t^2-1}dt$$

$$\frac{t^2}{t^2-1} = 1 + \frac{1}{t^2-1}$$

$$\int \frac{t^2}{t^2-1}dt = \int\left(1+\frac{1}{t^2-1}\right)dt = t + \int \frac{1}{t^2-1}dt$$

$$\frac{1}{t^2-1} = \frac{A}{t-1} + \frac{B}{t+1} = \frac{A(t+1)+B(t-1)}{t^2-1}$$

$$1 = A(t+1) + B(t-1)$$

$$t = 1 \to 1 = A\cdot 2 \to A = 1/2$$
$$t = -1 \to 1 = B\cdot(-2) \to B = -1/2$$

$$\int \frac{t}{t^2-1}\cdot dt = \frac{1}{2}\ln|t-1| - \frac{1}{2}\ln|t+1| = \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|$$

$$\int \frac{\sqrt{x}}{x-1}\cdot dx = 2\left(t + \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|\right) + C =$$

$$= 2\sqrt{x} + \ln\left|\frac{\sqrt{x}-1}{\sqrt{x}+1}\right| +$$

t^2 | t^2-1
$-t^2 - t^2$ | 1
$+1$

d) $\displaystyle\int \frac{12}{\sqrt{36-x^2}}\,dx$, $x = 6t$

$$\int \frac{12}{\sqrt{36-x^2}}dx = \int \frac{12}{\sqrt{36-36t^2}}6dt =$$
$$x = 6t \to dx = 6dt$$

$$= 12\int \frac{6}{6\sqrt{1-t^2}}dt = 12\int \frac{1}{\sqrt{1-t^2}}dt = 12\arcsin t +$$

$$+C = 12\arcsin\frac{x}{6} + C$$

13. Calcula la integral $\displaystyle\int \frac{7}{4+9x^2}\,dx$ mitjançant el canvi de variable $x = \frac{2}{3}t$. Aquesta integral, però, es quasi immediata. Calcula-la també sense fer canvi de variable.

$$\int \frac{7}{4+9x^2}dx = \frac{7}{4}\int \frac{1}{1+\left(\frac{3}{2}x\right)^2}dx =$$

$$= \frac{7}{4}\cdot\frac{2}{3}\int \frac{3/2}{1+\left(\frac{3}{2}x\right)}dx = \frac{7}{6}\operatorname{arctg}\frac{3x}{2} + C$$

$$x = \frac{2}{3}t \to dx = \frac{2}{3}dt$$

$$\int \frac{7}{4+9x^2}dx = 7\int \frac{1}{4+9\cdot\frac{4}{9}t^2}\cdot\frac{2}{3}dt =$$

$$= \frac{7}{4}\cdot\frac{2}{3}\int \frac{1}{1+t^2}dt = \frac{7}{6}\operatorname{arctg} t + C =$$

$$= \frac{7}{6}\operatorname{arctg}\frac{3x}{2} + C$$

14. Troba la primitiva de la funció $f(x) = \dfrac{x+2}{x-1}$ la gràfica de la qual passa pel punt (2, 2).

$$F(x) = \int \frac{x+2}{x-1}dx = \int\left(1+\frac{3}{x-1}\right)dx =$$

$$= x + 3\ln|x-1| + C$$
$$F(2) = 2$$

$$2 = 2 + 3\ln 1 + C \to C = 0$$
$$F(x) = x + 3\ln|x-1|$$

$x + 2$ | $x-1$
$-x + 1$ | 1
3

15. Calcula les integrals següents:

a) $\displaystyle\int \frac{5x^2}{2x^2-2}dx$

$$\int \frac{5x^2}{2x^2-2}dx = \frac{5}{2}\int \frac{x^2}{x^2-1}dx = \frac{5}{2}\int\left(1+\frac{1}{x^2-1}\right)dx$$

$$\frac{1}{x^2-1} = \frac{A}{x-1}+\frac{B}{x+1}$$

$$1 = A(x+1)+B(x-1)$$

$$x = 1 \rightarrow 1 = A\cdot 2 \rightarrow A = 1/2$$

$$x = -1 \rightarrow 1 = B\cdot(-2) \rightarrow B = -1/2$$

$$\frac{5}{2}\int\left(1+\frac{1}{x^2-1}\right)dx = \frac{5}{2}\left(x+\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|\right)+C$$

$$\begin{array}{l|l} 5x^2 & 2x^2-2 \\ \hline -5x^2+5 & \frac{5}{2} \\ \quad 5 & \end{array}$$

b) $\displaystyle\int \frac{\mathbf{4x-3}}{\mathbf{x^2-10x+25}}\mathbf{dx}$

$$\int\frac{4x-3}{x^2-10x+25}dx = \int\frac{4x-3-17+17}{x^2-10x+25}dx =$$

$$= \int\frac{4x-20}{x^2-10x+25}dx = 17\int\frac{1}{(x-5)^2}dx =$$

$$= 2\int\frac{2x-10}{x^2-10x+25}dx + 17\int(x-5)^{-2}dx =$$

$$= 2\ln\left|x^2-10x+25\right| - \frac{17}{x-5}+C$$

c) $\displaystyle\int \frac{\mathbf{x-1}}{\mathbf{x^3+2x^2-4x-8}}\mathbf{dx}$

$$\int\frac{x-1}{x^3+2x^2-4x-8}dx$$

$$x^3+2x^2-4x-8 = 0 \left\langle \begin{array}{ll} x = 2 & \text{(simple)} \\ x = -2 & \text{(doble)} \end{array}\right.$$

$$\frac{x-1}{x^3+2x^2-4x-8} = \frac{A}{x-2}+\frac{B}{x+2}+\frac{C}{(x+2)^2}$$

$$x-1 = A(x+2)^2+B(x-2)(x+2)+C(x-2)$$

$$x = 2 \rightarrow 1 = A\cdot 16 \rightarrow A = 1/16$$

$$x = -2 \rightarrow -3 = C\cdot(-4) \rightarrow C = 3/4$$

$$x = 0 \rightarrow -1 = A\cdot 4 + B\cdot(-4)+C\cdot(-2)$$

$$-1 = \frac{1}{4}-4B-\frac{3}{2} \rightarrow -4 = 1-16B-6 \rightarrow$$

$$\rightarrow B = -1/16$$

$$\int\frac{x-1}{x^3+2x^2-4x-8}dx = \frac{1}{16}\ln|x-2| -$$

$$-\frac{1}{16}\ln|x+2| - \frac{3}{4}\cdot\frac{1}{x+2}+C = \frac{1}{16}\ln\left|\frac{x-2}{x+2}\right| -$$

$$-\frac{3}{4(x+2)}+C$$

d) $\displaystyle\int \frac{\mathbf{x^2-1}}{\mathbf{x^2+4x}}\mathbf{dx}$

$$\int\frac{x^2-1}{x^2+4x}dx = \int\left(1+\frac{-4x-1}{x^2+4x}\right)dx =$$

$$= x + \int\frac{-4x-1}{x^2+4x}dx$$

$$\frac{-4x-1}{x^2+4x} = \frac{A}{x}+\frac{B}{x+4}$$

$$-4x-1 = A(x+4)+Bx$$

$$x = 0 \rightarrow -1 = A\cdot 4 \rightarrow A = -1/4$$

$$x = -4 \rightarrow 15 = B\cdot(-4) \rightarrow B = -15/4$$

$$\int\frac{x^2-1}{x^2+4x}dx = x - \frac{1}{4}\ln|x| - \frac{15}{4}\ln|x+4| + C$$

$$\begin{array}{l|l} x^2-1 & x^2+4x \\ \hline -x^2-4x & 1 \\ -4x-1 & \end{array}$$

e) $\displaystyle\int \frac{\mathbf{1}}{\mathbf{(x^2-x)(x+7)}}\mathbf{dx}$

$$\frac{1}{x(x-1)(x+7)} = \frac{A}{X}+\frac{B}{X+4}$$

$$1 = A(x-1)(x+7)+Bx(x+7)+Cx(x-1)$$

$$x = 0 \rightarrow 1 = A\cdot(-7) \rightarrow A = -1/7$$

$$x = 1 \rightarrow 1 = B\cdot 8 \rightarrow A = 1/8$$

$$x = -7 \rightarrow 1 = C\cdot 56 \rightarrow C = 1/56$$

f) $\displaystyle\int \frac{\mathbf{5-3x}}{\mathbf{3x-1}}\mathbf{dx}$

$$\int\frac{1}{(x^2-x)(x+7)}dx = -\frac{1}{7}\ln|x| + \frac{1}{8}\ln|x-1| +$$

$$+\frac{1}{56}\ln|x+7|+C$$

$$\int\frac{5-3x}{3x-1}dx = \int\left(-1+\frac{4}{3x-1}\right)dx = -x +$$

$$+\frac{4}{3}\ln|3x-1|+C$$

$$\begin{array}{l|l} -3x+5 & 3x-1 \\ \hline -3x-1 & -1 \\ \quad 4 & \end{array}$$

16. Calcula ∫sin² *x* d*x* i ∫cos² *x* d*x* a partir de les igualtats següents:

sin² *x* + cos² *x* = 1i cos 2*x* = cos² *x* – sin² *x*

$$\left.\begin{array}{l}\sin^2 x+\cos^2 x=1\\ -\sin^2 x+\cos^2 x=\cos 2x\end{array}\right\}\cos^2 x=\frac{1+\cos^2 x}{2};$$

$$\sin^2 x=\frac{1-\sin^2 x}{2}$$

$$\int\cos^2 x dx=\int\frac{1+\cos 2x}{2}dx=\frac{1}{2}x+\frac{\sin 2x}{4}+C$$

$$\int\sin^2 x dx=\int\frac{1-\cos 2x}{2}dx=\frac{1}{2}x-\frac{\sin 2x}{4}+C$$

17. Troba una primitiva de la funció següent:

$$f(x)=\frac{2x+1}{x^2+4}$$

Suggeriment: descompon la fracció en suma de dues fraccions del mateix denominador i fixa't en el canvi de variable utilitzat en l'exercici 13.

$$\int\frac{2x+1}{x^2+4}dx=\int\frac{2x}{x^2+4}dx+\int\frac{1}{x^2+4}dx=$$

$$=\ln(x^2+4)+\frac{1}{4}\cdot 2\int\frac{1/2}{(x/2)^2+1}dx=\ln(x^2+4)+$$

$$+\frac{1}{2}\text{arctg}\frac{x}{2}+C$$

Com que ens demanen una primitiva, fem, per exemple, C=0.

18. Un mòbil es desplaça sobre l'eix *OX* de manera que la seva acceleració ve donada per l'equació següent:

$a = 2t + 1$ m/s²

Si per a $t = 0$ es verifica $v(0) = -2$ m/s i $x(0) = 10$ m, troba les expressions de les funcions velocitat $v = v(t)$ i posició $x = x(t)$ corresponents a aquest mòbil.

$$v=\int(2t+1)dx=t^2+t+C$$

$$v(0)=-2\rightarrow C=-2$$

$$v=t^2+t-2\frac{\text{m}}{\text{s}}$$

$$x=\int(t^2+t-2)dx=\frac{t^3}{3}+\frac{t^2}{2}-2t+C'$$

$$x(0)=10\rightarrow C'=10$$

$$x=\frac{1}{3}t^3+\frac{1}{2}t^2-2t+10\ (\text{m})$$

19. Un mòbil descriu un moviment vibratori harmònic simple l'acceleració del qual s'expressa per l'equació $a = -36\cos 3t$ cm/s².

Si a l'instant inicial es verifica $v(0) = 0$ cm/s i $x(0) = 4$ cm, troba les expressions de les funcions velocitat $v = v(t)$ i posició $x = x(t)$ d'aquest mòbil.

$$v=\int-36\cos 3t dt=-12\int 3\cos 3t dt=$$

$$=-12\sin 3t+C$$

$$v(0)=0\rightarrow C=0\rightarrow v=-12\sin 3t\left(\frac{\text{cm}}{\text{s}}\right)$$

$$x=\int-12\sin 3t dt=4\int-3\sin 3t dt=4\cos 3t+C'$$

$$x(0)=4\rightarrow 4=4+C'\rightarrow C'=0$$

$$x=4\cos 3t\ (\text{cm})$$

20. Calcula $\int\frac{dx}{\sqrt{x+1}+\sqrt{x-1}}$.

Indicació: multiplica primer numerador i denominador per l'expressió conjugada del denominador.

$$\frac{1}{\sqrt{x+1}+\sqrt{x-1}}=\frac{1}{\sqrt{x+1}+\sqrt{x-1}}\cdot$$

$$\cdot\frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}=\frac{\sqrt{x+1}-\sqrt{x-1}}{2}$$

$$\int\frac{dx}{\sqrt{x+1}+\sqrt{x-1}}=\frac{1}{2}\int(\sqrt{x+1}-\sqrt{x-1})dx=$$

$$=\frac{1}{2}\left[\frac{(x+1)^{3/2}}{\frac{3}{2}}-\frac{(x-1)^{3/2}}{\frac{3}{2}}\right]+C=$$

$$=\frac{1}{3}\left[\sqrt{(x+1)^3}-\sqrt{(x-1)^3}\right]+C$$

21. Calcula $\int\frac{dx}{1-\sin x}$.

Indicació: multiplica primer numerador i denominador per $1 + \sin x$.

$$\frac{1}{1-\sin x}=\frac{1}{1-\sin x}\cdot\frac{1+\sin x}{1+\sin x}=\frac{1-\sin x}{\cos^2 x}$$

$$\int\frac{1}{1-\sin x}dx=\int\left(\frac{1}{\cos^2 x}+\frac{\sin x}{\cos^2 x}\right)dx=$$

$$=\text{tg}x+\frac{1}{\cos x}+C$$

22. Troba la primitiva de la funció

$$f(x)=\frac{-1}{(x-2)^2}$$

la gràfica de la qual té per asímptota horitzontal la recta y = 2.

$$\int \frac{-1}{(x-2)^2}dx = \int -(x-2)^{-2}dx = \frac{1}{x-2} + C$$

$$\lim_{x\to\infty}\left(\frac{1}{x-2} + C\right) = 2 \to C = 2 \to \frac{1}{x-2} + 2 = \frac{2x-3}{x-2}$$

23. Calcula les integrals següents:

***a*) $\int (\text{tg}^5 x + \text{tg}^7 x)\, dx$**

$$\int (\text{tg}^5 x + \text{tg}^7 x)dx = \int \text{tg}^5 x(1+\text{tg}^2 x)dx =$$

$$= \frac{\text{tg}^6 x}{6} + C$$

***b*) $\int \frac{dx}{\sqrt{x}\cos^2\sqrt{x}}$**

$$\int \frac{dx}{\sqrt{x}\cos^2\sqrt{x}} = 2\int \frac{1}{2\sqrt{x}}\cdot\frac{1}{\cos^2\sqrt{x}}dx =$$

$$= 2\text{tg}\sqrt{x} + C$$

***c*) $\int \frac{1}{(x-1)(x^2+5x+6)}dx$**

$$\int \frac{1}{(x-1)(x^2+5x+6)}dx =$$

$$= \int \frac{1}{(x-1)(x+2)(x+3)}dx$$

$$1 = A(x+2)(x+3) + B(x-1)(x+3) + $$
$$+ C(x-1)(x+2)$$

$$x = 1 \to 1 = A\cdot 12 \to A = 1/12$$
$$x = -2 \to 1 = A\cdot(-3) \to B = -1/3$$
$$x = -3 \to 1 = C\cdot 4 \to C = 1/4$$

$$\int \frac{1}{(x-1)(x+2)(x+3)}dx = \frac{1}{2}\ln|x-1| -$$
$$-\frac{1}{3}\ln|x+2| + \frac{1}{4}\ln|x+3| + C$$

***d*) $\int \frac{3^x}{1+3^{2x}}dx$**

$$\int \frac{3^x}{1+3x^{2x}}dx = \frac{1}{\ln 3}\int \frac{3^x\cdot\ln 3}{1+(3^x)^2} = \frac{\text{arctg}3^x}{\ln 3}$$

***e*) $\int \frac{dx}{\cos^2 x(1+2\text{tg}x)}$**

$$\int \frac{dx}{\cos^2(1+2\text{tg}x)} = \frac{1}{2}\int \frac{2\cdot\frac{1}{\cos^2 x}}{1+2\text{tg}x}dx =$$

$$= \frac{1}{2}\ln|1+2\text{tg}x| + C$$

***f*) $\int \frac{x^4}{1+x^2}dx$**

$$\int \frac{x^4}{1+x^2}dx = \int\left(x^2 - 1 + \frac{1}{x^2+1}\right)dx =$$

$$= \frac{x^3}{3} - x + \text{arctg}x + C$$

$x4$ | $x2 + 1$
$-x4 - x2$ | $x2 - 1$
$-x2$
$-x2 + 1$
1

***g*) $\int \cos x\sqrt[5]{7+\sin x}\,dx$**

$$\int \cos x\sqrt[5]{7+\sin x}\,dx = \int \cos x(7+\sin x)^{1/5}dx =$$

$$= \frac{(7+\sin x)^{6/5}}{\frac{6}{5}} + C = \frac{5}{6}\sqrt[5]{(7+\sin x)^6} + C$$

***h*) $\int \sqrt{\frac{\arcsin x}{1-x^2}}dx$**

$$\int \sqrt{\frac{\arcsin x}{1-x^2}}dx = \int (\arcsin x)^{1/2}\cdot\frac{1}{\sqrt{1-x^2}}dx =$$

$$= \frac{(\arcsin x)^{3/2}}{3/2} + C = \frac{2}{3}\sqrt{(\arcsin x)^3} + C$$

24. Determina la primitiva de la funció

$$f(x) = \frac{2}{x-3}$$

la gràfica de la qual conté el punt (4, 0). Anomena $F(x)$ aquesta funció i calcula $\lim_{x\to 3^+} F(x)$ i $\lim_{x\to 3^-} F(x)$.

Dibuixa de manera aproximada la gràfica de la funció $F(x)$.

$$F(x) = \int \frac{2}{x-3}dx = 2\ln|x-3| + C$$
$$F(4) = 0 \to 0 = 2\ln 1 + C \to C = 0$$
$$F(x) = 2\ln|x-3| = \ln(x-3)^2$$
$$\lim_{x\to\infty} F(x) = \lim_{x\to 3^-} F(x) = -\infty$$

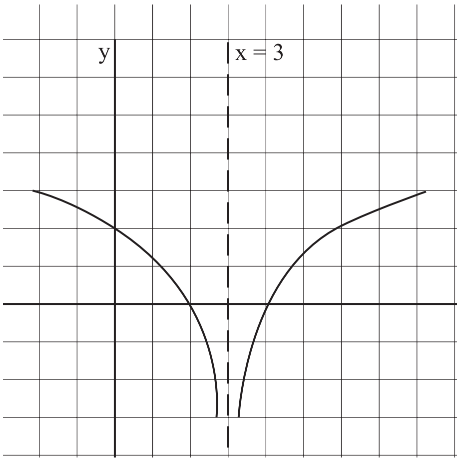


Fig. 5.3

25. Troba l'expressió algèbrica de la funció *F*(*x*) que verifica les condicions següents:

***a*) *F*'(*x*) = 2*x* – 6**

$$F(x) = \int (2x-6)dx = x^2 - 6x + C$$

***b*) La gràfica de la funció *F*(*x*) presenta un mínim en el punt d'ordenada –1.**

Mínim $\rightarrow F'(x) = 0 \rightarrow 2x - 6 = 0 \rightarrow x = 3$

$F(3) = -1 \rightarrow -1 = 9 - 18 + C \rightarrow C = 8$

$F(x) = x^2 - 6x + 8$

26. Aplicant el mètode d'integració per parts, calcula $\int \cos^2 x\, dx$.

Cal que tinguis en compte que $\cos^2 x = \cos x \cos x$ i que $\sin^2 x = 1 - \cos^2 x$. Compara el resultat amb el que has obtingut a l'exercici 16.

$$\int \cos^2 x dx = \int \cos x \cdot \cos x dx$$

$$f(x) = \cos x \rightarrow f'(x) = -\sin x$$

$$g'(x) = \cos x \rightarrow g(x) = \sin x$$

$$\int \cos^2 x dx = \sin\cos x + \int \sin^2 x dx = \sin\cos x +$$

$$= \int (1-\cos^2 x)dx = \sin x \cos x + x - \int \cos^2 x dx$$

$$2\int \cos^2 x dx = x + \sin x \cos x$$

$$\int \cos^2 x dx = \frac{x}{2} + \frac{\sin x \cos x}{2} + C =$$

$$= \frac{x}{2} + \frac{\sin x \cos x}{4} + C = \frac{x}{2} + \frac{\sin 2x}{4} + C$$